Converting Repeating Decimals to Rational Numbers

(Remember “rational numbers” is just a more sophisticated mathematical way of talking about fractions).

When we talked about the Real Number system, Rational and Irrational numbers, we said Irrational numbers are decimals that literally continue forever, and that they never form a pattern nor do they converge to a repeating number such as \(0.\overline{6}\).

The purpose of this lesson is to learn how to convert any repeating decimal, whether a repeating single digit or a repeating series, into a rational number or a fraction.

Consider \(0.\overline{1}\)

The first step is always to establish an equation by seeing the decimal in question equal to \(x\).

Let \(x = 0.\overline{1}\)

Of course we know we can multiply both sides of an equal sign by the same thing and both sides will remain equal.

Multiply both sides by 10: (Notice the effect on moving the decimal.)

\[10x = 1.\overline{1}\]

From the first equation, \(x = 0.\overline{1}\), both sides are equal so I can subtract the first equation from the second equation for the following:

\[10x = 1.\overline{1}\]

\[-(x = 0.\overline{1})\]

Subtracting \(x\) from 10\(x\) is 9\(x\), and on the right side of the equal sign the \(\overline{1}\) is subtracted from both equations leaving just the 1.

The answer is:
\[ 9x = 1 \text{ or } x = \frac{1}{9} \]

**Second Example:**

Let's look at 0.\text{\overline{6}}

The process is always the same.

First make an equation by setting \( x \) equal to the repeating decimal in question. Then it is only a matter of what powers of 10 you will multiply one or both of the equations by such that the repeating parts will subtract out and you can solve the equations of \( x \).

\[ x = 0.\overline{6} \]

Multiply by sides by 10 for the second equation: \( 10x = 6.\overline{6} \)

Subtract the first equation from the second:

\[ 10x = 6.\overline{6} \]

\[ -(x = 0.\overline{6}) \]

\[ 9x = 6 \text{ or } x = \frac{6}{9} = \frac{2}{3} \]

Now a slightly more difficult series: \( x = 3.\overline{4} \)

Note that if I simply multiply the equation by 10 I’m only moving the decimal one place, and the repeating part will not subtract out. This is caused by the non-repeating 3.

In this case I’ll multiply both equations such that the repeating part subtracts out.

\[ x = 0.3\overline{4} \]

Multiplying by 10 yields \( 10x = 3.\overline{4} \) and multiplying by 100 yields,

\[ 100x = 34.\overline{4} \]
The subtractions gives us:

\[ 100x = 34.\bar{4} \]

\[-(10x = 3.4) \]

\[ 90x = 31 \text{ or } x = \frac{31}{90} \]

**Third Example:**

Here’s one that looks pretty hard: 0.\overline{5786}

\[ x = .\overline{5786} \text{ and } 10,000x = 5786.\overline{5786} \]

\[ 10,000x = 5786.\overline{5786} \]

\[-(x = .(5786)) \]

\[ 9,999x = 5686 \text{ or } x = \frac{5786}{9,999} \]

**Fourth Example:**

0.\overline{3752}

Again, the main issue is determining how many powers of 10 you will multiply the equations by such that the repeating part subtracts out.

Multiplying by 100 moves the decimal to right in front of the repeating part. Then I have to multiply it again by a power of 10 large enough that the repeating part will subtract out.

\[ x = .\overline{3752} \]

\[(100x = 37.\overline{52})\]

\[ 10,000x = 3752.\overline{52} \]
Consider the subtraction:

\[ 10,000x = 3752.52 \]

\[-(100x = 37.52)\]

\[ 9,900x = 3715 \text{ or } x = \frac{3,715}{9,900} \]